

# ON SOME INEQUALITIES OF CHEBYSHEV TYPE

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*Abstract.* We obtain some new inequalities of Chebyshev Type.

## 1. Introduction.

Let  $f, g: [a, b] \rightarrow \mathbb{R}$  be integrable functions, both increasing or both decreasing. Further, let  $p: [a, b] \rightarrow \mathbb{R}_0^+$  be an integrable function. Then (see, for example, [1, Chap. IX])

$$\int_a^b p(x)f(x)g(x)dx \geq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \left( \int_a^b p(x)dx \right)^{-1}. \quad (1)$$

If one of the functions  $f$  or  $g$  is nonincreasing and the other nondecreasing the reversed inequality is true, i.e.,

$$\int_a^b p(x)f(x)g(x)dx \leq \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx \left( \int_a^b p(x)dx \right)^{-1}. \quad (2)$$

Inequalities (1) and (2) are known as Chebyshev's inequalities. These inequalities were obtained by P.L. Chebyshev [2, 3] and they attracted great interest of the researchers. So, a lot of analogues and generalizations of inequalities (1) and (2) is known. In particular, these results can be found in Chapter IX of the book [1] by D.S. Mitrinović, J.E. Pečarić and A.M. Fink which trace completely the historical and chronological developments of Chebyshev's and related inequalities (see also [4, 5]). Also we would like to recommend the article of H.P. Heinig and L. Maligranda [6], where one can found a lot of important results on Chebyshev's inequalities for strongly increasing functions, positive convex and concave functions as well as on Chebyshev's inequalities in Banach function spaces and symmetric spaces.

In [7], these investigations were developed in the following direction: the author found necessary and sufficient conditions on the function  $g: [a, b] \rightarrow \mathbb{R}_0^+$  and  $p: [a, b] \rightarrow \mathbb{R}^+$  such that for any monotone function  $f: [a, b] \rightarrow \mathbb{R}_0^+$  the relations

$$\int_a^b p(x)f(x)g(x)dx \geq \left( \int_a^b p^r(x)f^r(x)dx \right)^{1/r} \int_a^b p(x)g(x)dx \left( \int_a^b p^r(x)dx \right)^{-1/r} \quad (3)$$

and

$$\int_a^b p(x)f(x)g(x)dx \leq \left( \int_a^b p^r(x)f^r(x)dx \right)^{1/r} \int_a^b p(x)g(x)dx \left( \int_a^b p^r(x)dx \right)^{-1/r} \quad (4)$$

hold with  $r$  being an arbitrary positive number.

In this paper we continue the study of the inequalities of the type (1)–(4), namely, we obtain the following assertions:

**THEOREM 1.** Assume that  $g: [a, b] \rightarrow \mathbb{R}_0^+$  and  $p: [a, b] \rightarrow \mathbb{R}^+$  are integrable functions such that the product  $p \cdot g$  is also integrable on  $[a, b]$  function. Let  $f: [a, b] \rightarrow \mathbb{R}_0^+$  be a nonincreasing function. Then for any convex function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , the following inequality is true:

$$\int_a^b p(x)g(x)M(f(x))dx \leq \sup_{s \in (a, b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\}, \quad (5)$$

and for any concave function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , the following inequality is true:

$$\int_a^b p(x)g(x)M(f(x))dx \geq \inf_{s \in (a, b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\}. \quad (6)$$

Furthermore, if the function  $f(x) \equiv c$ ,  $c \geq 0$ , then relations (5) and (6) are equalities.

Putting  $M(t) = t^{1/r}$ ,  $r > 0$ , from Theorem 1 we obtain the following corollaries:

**COROLLARY 1.** Let  $r \in (0, 1]$ , and let  $g: [a, b] \rightarrow \mathbb{R}_0^+$  and  $p: [a, b] \rightarrow \mathbb{R}^+$  be integrable functions such that for all  $s \in (a, b]$ ,

$$\frac{\int_a^s p(x)g(x)dx}{\left(\int_a^s p(x)dx\right)^{1/r}} \leq \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \quad (7)$$

Then for any nonincreasing function  $f: [a, b] \rightarrow \mathbb{R}_0^+$ ,

$$\int_a^b p(x)g(x)f(x)dx \leq \left(\int_a^b p(x)f^r(x)dx\right)^{1/r} \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \quad (8)$$

**COROLLARY 2.** Let  $r \geq 1$ , and let  $g: [a, b] \rightarrow \mathbb{R}_0^+$  and  $p: [a, b] \rightarrow \mathbb{R}^+$  be integrable functions such that for all  $s \in (a, b]$ ,

$$\frac{\int_a^s p(x)g(x)dx}{\left(\int_a^s p(x)dx\right)^{1/r}} \geq \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \quad (9)$$

Then for any nonincreasing function  $f: [a, b] \rightarrow \mathbb{R}_0^+$ ,

$$\int_a^b p(x)g(x)f(x)dx \geq \left(\int_a^b p(x)f^r(x)dx\right)^{1/r} \frac{\int_a^b p(x)g(x)dx}{\left(\int_a^b p(x)dx\right)^{1/r}}. \quad (10)$$

If in corollaries 1 and 2, we put  $r = 1$ , then we see that relations (8) and (10) are the Chebyshev's classical inequalities (1) and (2). Furthermore, it should be noted that conditions on the functions  $p$  and  $g$  of the form (7) and (9) for validity of inequalities (1) and (2) were considered in the papers [7] and [8].

In the case, where the function  $M(f(x))$  is nonincreasing and the function  $g$  is nondecreasing (or nonincreasing), we can apply the Chebyshev's classical inequalities to the integral  $\int_a^b p(x)g(x)M(f(x))dx$  on the left-hand side of relations (5) (or (6)). Respectively, we obtain

$$\int_a^b p(x)g(x)M(f(x))dx \leq \frac{\int_a^b p(x)M(f(x))dx}{\int_a^b p(x)dx} \int_a^b p(x)g(x)dx \quad (11)$$

and

$$\int_a^b p(x)g(x)M(f(x))dx \geq \frac{\int_a^b p(x)M(f(x))dx}{\int_a^b p(x)dx} \int_a^b p(x)g(x)dx. \quad (12)$$

Furthermore, if exact upper (or lower) bound on the right-hand side of (5) (or (6)) is realized for  $s = b$ , then from relations (5) and (6) we get

$$\int_a^b p(x)g(x)M(f(x))dx \leq M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}\right) \int_a^b p(x)g(x)dx, \quad (13)$$

and

$$\int_a^b p(x)g(x)M(f(x))dx \geq M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}\right) \int_a^b p(x)g(x)dx. \quad (14)$$

Here, it should be note that by virtue of Jensen's inequality (see, for example, [1, Chap. I]), estimations (13) and (14) of the integral  $\int_a^b p(x)g(x)M(f(x))dx$  are more precisely, than estimations (11) and (12).

REMARK. In the case, where the function  $f$  does not decrease, inequalities (5) and (6) have the similar form, but in these inequalities, all the integrals of the kind  $\int_a^s(\cdot)$  should be replaced by the integrals of the kind  $\int_s^b(\cdot)$ .

## 2. Discrete analogue of Theorem 1.

LEMMA 1. Assume that  $a = \{a_k\}_{k=1}^m$ ,  $b = \{b_k\}_{k=1}^m$  and  $p = \{p_k\}_{k=1}^m$ ,  $m \in \mathbb{N}$  are nonnegative number sequences such that  $a_1 \geq a_2 \geq \dots a_m$  and  $p_k > 0$ . Then for any convex function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , the following inequality is true:

$$\sum_{k=1}^m p_k b_k M(a_k) \leq \max_{s \in [1, m]} \left\{ M\left(\frac{\sum_{k=1}^m p_k a_k}{\sum_{k=1}^s p_k}\right) \sum_{k=1}^s p_k b_k \right\}, \quad (15)$$

and for any concave function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , the following inequality is true:

$$\sum_{k=1}^m p_k b_k M(a_k) \geq \min_{s \in [1, m]} \left\{ M\left(\frac{\sum_{k=1}^m p_k a_k}{\sum_{k=1}^s p_k}\right) \sum_{k=1}^s p_k b_k \right\}. \quad (16)$$

Furthermore, if the sequence  $a_k \equiv c$ ,  $c \geq 0$ , then relations (15) and (16) are equalities.

*Proof.* Consider the case, where the function  $M$  is convex (in the case, where the function  $M$  is concave, the proof is similar). Let us prove by the induction on  $m$  the proposition that for any convex function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , inequality (15) holds.

The case  $m = 1$  is obvious.

Also consider the case  $m = 2$ .

Put

$$c = p_1 a_1 + p_2 a_2, \quad x_0 = p_1 a_1, \quad \alpha_k = p_k b_k, \quad \beta_k = p_k^{-1}, \quad k = 1, 2, \quad (17)$$

and consider on the interval  $[0, c]$  the function

$$h(x) = \alpha_1 M(\beta_1 x) + \alpha_2 M(\beta_2 (c - x)). \quad (18)$$

Due to convexity of the function  $M(t)$ , the function  $h(x)$  is also convex on  $[0, c]$ . Hence, this function attains its maximum value on any interval  $[x_1, x_2] \subseteq [0, c]$  at one of its endpoints. Thus

$$h(x) \leq \max\{h(x_1), h(x_2)\} \quad \forall x \in [x_1, x_2]. \quad (19)$$

Setting  $x_1 := \beta_2 c(\beta_1 + \beta_2)^{-1}$  and  $x_2 := c$ , we see that the number  $x_0$  (by virtue of monotonicity of the sequence  $a$ ) belongs to the interval  $[x_1, x_2]$ .

Therefore, in view of relations (17)–(19) and of the equality  $M(0) = 0$ , we get

$$\begin{aligned} \sum_{k=1}^2 p_k b_k M(a_k) &= h(x_0) \leq \max\{h(x_1), h(x_2)\} = \\ &= \max \left\{ M\left(\frac{p_1 a_2 + p_2 a_2}{p_1 + p_2}\right)(p_1 b_2 + p_2 b_2), M\left(\frac{p_1 a_2 + p_2 a_2}{p_1}\right)p_1 b_1 \right\}. \end{aligned}$$

Hence, for  $m = 2$ , inequality (15) holds.

Now, assume that for  $m = n - 1 \geq 1$ , the proposition is true.

Let us show that for  $m = n$ , it is also true. Let us use notations (17) and consider on the interval  $[0, c]$  the function  $h(x)$  of the form as in (18). Setting  $x_1 := \beta_2 c(\beta_1 + \beta_2)^{-1}$  and  $x_2 := c - a_3/\beta_2$ , we see that the number  $x_0$  (by virtue of monotonicity of the sequence  $a$ ) belongs to the interval  $[x_1, x_2]$ . Thus in view of relations (17)–(19),

$$\sum_{k=1}^n p_k b_k M(a_k) = h(x_0) + \sum_{k=3}^n p_k b_k M(a_k) \leq \max\{h(x_1), h(x_2)\} + \sum_{k=3}^n p_k b_k M(a_k). \quad (20)$$

Further, in the case, where  $h(x_1) \geq h(x_2)$ , we set

$$p'_k = \begin{cases} p_1 + p_2, & k = 1, \\ p_{k+1}, & k = \overline{2, m-1}; \end{cases} \quad b'_k = \begin{cases} (p_1 b_1 + p_2 b_2)/(p_1 + p_2), & k = 1, \\ b_{k+1}, & k = \overline{2, m-1}; \end{cases} \quad (21)$$

$$a'_k = \begin{cases} (p_1 a_1 + p_2 a_2)/(p_1 + p_2), & k = 1, \\ a_{k+1}, & k = \overline{2, m-1}. \end{cases} \quad (22)$$

Then by virtue of (20), we conclude that the following relation is true:

$$\sum_{k=1}^m p_k b_k M(a_k) \leq \sum_{k=1}^{m-1} p'_k b'_k M(a'_k). \quad (23)$$

In the case, where  $h(x_1) < h(x_2)$ , relation (23) holds for the sequences  $a'$ ,  $b'$  and  $p'$  of the form:

$$p'_k = \begin{cases} p_1, & k = 1, \\ p_2 + p_3, & k = 2, \\ p_{k+1}, & k = \overline{3, m-1}; \end{cases} \quad b'_k = \begin{cases} b_1, & k = 1, \\ (p_2 b_2 + p_3 b_3)(p_2 + p_3)^{-1}, & k = 2, \\ b_{k+1}, & k = \overline{3, m-1}; \end{cases} \quad (24)$$

$$a'_k = \begin{cases} (p_1 a_1 + p_2 a_2 - p_2 a_3)/p_1, & k = 1, \\ a_{k+1}, & k = \overline{2, m-1}, \end{cases} \quad (25)$$

The sum on the right-hand side of (23) contains  $n - 1$  items. Furthermore, in both cases, for the sequences  $a'$ ,  $b'$  and  $p'$ , the induction assumption is satisfied. Thus, taking into account (21)–(25), we obtain the necessary estimate (15):

$$\begin{aligned} \sum_{k=1}^n p_k b_k M(a_k) &\leq \sum_{k=1}^{n-1} p'_k b'_k M(a'_k) \leq \sup_{s \in [1, n-1]} \left\{ M \left( \frac{\sum_{k=1}^{n-1} p'_k a'_k}{\sum_{k=1}^s p'_k} \right) \sum_{k=1}^s p'_k b'_k \right\} \leq \\ &\leq \sup_{s \in [1, n]} \left\{ M \left( \frac{\sum_{k=1}^n p_k a_k}{\sum_{k=1}^s p_k} \right) \sum_{k=1}^s p_k b_k \right\}. \end{aligned}$$

Lemma is proved.

### 3. Proof of Theorem 1.

Consider the case, where the function  $M$  is convex (in the case, where the function  $M$  is concave, the proof is similar). First, let us verify that inequality (5) holds for any function  $f$  such that for a certain  $m \in \mathbb{N}$ ,

$$f(x) = a_k, \quad x \in [l_{k-1}, l_k), \quad k = 1, 2, \dots, m,$$

where  $a_1 > a_2 > \dots > a_m \geq 0$  and  $a = l_0 < l_1 < \dots < l_m = b$ .

For any  $k = 1, 2, \dots, m$ , we put

$$p_k = \int_{l_{k-1}}^{l_k} p(x) dx, \quad b_k = \int_{l_{k-1}}^{l_k} p(x) g(x) dx \left( \int_{l_{k-1}}^{l_k} p(x) dx \right)^{-1}.$$

Then by virtue of Lemma 1, we get (5):

$$\begin{aligned} \int_a^b p(x) g(x) M(f(x)) dx &= \sum_{k=1}^m \int_{l_{k-1}}^{l_k} p(x) g(x) M(f(x)) dx = \sum_{k=1}^m p_k b_k M(a_k) \leq \\ &\leq \sup_{s \in [1, m] \cap \mathbb{N}} \left\{ M \left( \frac{\sum_{k=1}^m p_k a_k}{\sum_{k=1}^s p_k} \right) \sum_{k=1}^s p_k b_k \right\} = \sup_{s \in [1, m] \cap \mathbb{N}} \left\{ M \left( \frac{\int_a^b p(x) f(x) dx}{\int_a^{l_s} p(x) dx} \right) \int_a^{l_s} p(x) g(x) dx \right\} \leq \\ &\leq \sup_{s \in (a, b]} \left\{ M \left( \frac{\int_a^b p(x) f(x) dx}{\int_a^s p(x) dx} \right) \int_a^s p(x) g(x) dx \right\}. \end{aligned}$$

Let us prove the validity of inequality (5) in general case. First, note that if the functions  $M$  and  $f$  satisfy the conditions of Theorem 1, then there exists the number  $n_0 = n_0(M, f) \in \mathbb{N}$  such that for any  $n > n_0$  and for all  $x \in [a, b]$ , the inequality  $|M(f(x))| < n$  holds.

For any  $n > n_0$ , consider the system of points  $l_0^{(n)} < l_1^{(n)} < \dots < l_m^{(n)} = b$ , defined in the following way: we put  $l_0^{(n)} := a$  and for any  $k \in [1, m] \cap \mathbb{N}$  the value  $l_k^{(n)}$  is a greatest positive number such that  $l_k^{(n)} > l_{k-1}^{(n)}$  and for all  $x \in [l_{k-1}^{(n)}, l_k^{(n)})$  the following relation is true:

$$|M(f(l_{k-1}^{(n)})) - M(f(x))| \leq \frac{1}{n}.$$

By virtue of the conditions on the function  $M$  and  $f$ , this system of points always exist and  $m \leq 2n^2$ .

Further, consider the functions  $f_n = f_n(x)$  such that

$$f_n(x) \equiv \lim_{t \rightarrow l_k^{(n)} -} f(t), \quad \text{for all } x \in [l_{k-1}^{(n)}; l_k^{(n)}), \quad k = 1, 2, \dots, m. \quad (26)$$

We see that the inequality  $|M(f(x)) - M(f_n(x))| \leq \frac{1}{n}$  holds for all  $n > n_0$  and  $x \in [a, b]$ . Due to integrability on  $[a, b]$  of the product  $p(x)g(x)$ , the values

$$\int_a^b p(x)g(x)[M(f(x)) - M(f_n(x))] dx$$

converge to zero as  $n \rightarrow \infty$ . Furthermore, for any  $n > n_0$ , the function  $f_n(x)$  is nonincreasing and it takes finitely many values on  $[a, b]$ . Hence, this function satisfies the conditions of the proposition proved above.

Thus, in view of (26) and continuity of the function  $M$ , we conclude that for any  $\varepsilon > 0$  and for all sufficiently great  $n$  ( $n > n_1(\varepsilon)$ ),

$$\begin{aligned} \int_a^b p(x)g(x)M(f(x))dx &= \int_a^b p(x)g(x)M(f_n(x))dx + \int_a^b p(x)g(x)(M(f(x)) - M(f_n(x)))dx \leq \\ &\leq \sup_{s \in (a; b]} \left\{ M\left(\frac{\int_a^b p(x)f_n(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\} + \frac{\varepsilon}{2} \leq \sup_{s \in (a; b]} \left\{ M\left(\frac{\int_a^b p(x)f(x)dx}{\int_a^s p(x)dx}\right) \int_a^s p(x)g(x)dx \right\} + \varepsilon. \end{aligned}$$

Hence, relation (5) is true.

Analyzing the proof of Theorem 1, we see that the similar statement is also true in the case, where  $b = \infty$ .

**THEOREM 1'.** Assume that  $g: [a, b] \rightarrow \mathbb{R}_0^+$  and  $p: [a, b] \rightarrow \mathbb{R}^+$  (where  $b \in (a, \infty]$ ) are integrable functions such that the product  $p \cdot g$  is also integrable on  $[a, b]$  function. Let also  $f: [a, b] \rightarrow \mathbb{R}_0^+$  be a nonincreasing function. Then for any convex (or concave) function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , inequality (5) (or inequality (6)) is true.

Analogically, one can obtain the statement, similar to Lemma 1, in the case, where  $n = \infty$ .

**LEMMA 1'.** Let  $a = \{a_k\}_{k=1}^\infty$ ,  $b = \{b_k\}_{k=1}^\infty$  and  $p = \{p_k\}_{k=1}^\infty$  be nonnegative number sequences such that  $a_1 \geq a_2 \geq \dots$ ,  $p_k > 0$  and the series  $\sum_{k=1}^\infty p_k b_k$  is convergent. Then for any convex function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , the following inequality is true:

$$\sum_{k=1}^\infty p_k b_k M(a_k) \leq \sup_{s \in [1, \infty)} \left\{ M\left(\frac{\sum_{k=1}^\infty p_k a_k}{\sum_{k=1}^s p_k}\right) \sum_{k=1}^s p_k b_k \right\}, \quad (15')$$

and for any concave function  $M: [0, \infty) \rightarrow \mathbb{R}$  such that  $M(0) = 0$ , the following inequality is true:

$$\sum_{k=1}^\infty p_k b_k M(a_k) \geq \inf_{s \in [1, \infty)} \left\{ M\left(\frac{\sum_{k=1}^\infty p_k a_k}{\sum_{k=1}^s p_k}\right) \sum_{k=1}^s p_k b_k \right\}. \quad (16')$$

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